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CHIELLINI INTEGRABILITY CONDITION AND NEW INTEGRABLE SYSTEMS

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ABSTRACT

We rejuvenate the Chiellini integrability method to generate some new nonlinear integrable equations and find their analytic solutions. A generalised class of Van der Pol equation with a new form of potential, a damped Ermakov-Painlevé II system and a generalised Milne-Pinney equation is analysed.

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KEYWORDS: Chiellini Condition, Milne-Ermakov-Pinney Equation, Generalised Van der Pol equation & Weierstrass $\wp$ ***Function.***

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Original Article

1. INTRODUCTION

Ermakov [1] used the Milne Pinney equation [2], [3] while investigating a first integral for the corresponding time dependent harmonic oscillator. Since then, this nonlinear equation has gained intensive attention [4], [5] of physicists and engineers due to its widespread application in many physical problems such as propagation of laser beams in nonlinear media, plasma dynamics etc. It is well known that the general solution for the Milne-Ermakov-Pinney equation

$$\ddot{y}+\omega^2(t)y=\frac{\kappa}{y^3}, \tag{1.1}$$

where $\ddot{y}$ denotes double differentiation of y with respect to time t, $\omega=\omega(t)$ is a time-dependent frequency function and κ is a numerical constant can be written as $y=(Ax_1^2+2Bx_1x_2+Cx_2^2)^{1/2}$, where A, B and C are constants such that $AC-B^2=\kappa$ and x_1 and x_2 are two independent solutions for the time-dependent harmonic oscillator equation $\ddot{x}+\omega^2(t)x=0$.

Equation (1) does not include any mechanism of damping. Hence it is natural to add a term linear in the velocity, yielding the damped Milne-Ermakov-Pinney equation

$$\ddot{y}+\mu\dot{y}+\omega^2(t)y=\frac{\kappa}{y^3}, \tag{1.2}$$

where $\mu > 0$ is a constant positive parameter. Equation (1.2) can be transformed into generalized Emden-Fowler equation of index -3 which satisfies integrability.

In recent times a hybrid Ermakov-Painlevé II system was derived by Rogers [6] in a pioneering work as a reduction of a coupled $N+1$-dimensional Manakov-type NLS system. He showed that the Ermakov invariants admitted by the hybrid system were key to its systematic reduction in terms of a single component Ermakov-Painlevé II equation which, in turn, may be linked to the integrable Painlevé II equation.

The force free Van der Pol equation and the Duffing equation are characterised by the equation $\ddot{x} = g(x,\dot{x},t) + \gamma(t)$ with $g(x,\dot{x},t)$ and $\gamma(t)$ in the form

$$g(x,\dot{x},t) = (\alpha - \beta x^2)\dot{x} - \frac{dV}{dx}, \qquad \gamma(t) = 0, \tag{1.3}$$

where V is the potential function approximated by a finite Taylor's expansion in series $V_2(x) = \frac{1}{2}\omega_0^2 x^2$ and $V_4(x) = \frac{1}{2}\omega_0^2 x^2 + \frac{1}{4}\alpha_0 x^4$ respectively, ω_0 and λ_0 are non zero and α_0 is a real number.

The Van der Pol equation introduced by Dutch electrical engineer Balthazar Van der Pol in 1927, has gained considerable importance due to its usefulness in modeling various physical systems. Since its introduction, it has been a prototype for oscillatory systems in various fields of science. For $\mu = 0$ the equation reduces to the well known simple harmonic motion. But for $\mu \neq 0$ the term $\mu(1-x^2)\dot{x}$ plays an important role in describing nonlinear dissipative systems.

The Duffing equation, named after George Duffing in 1918 attracted the attention of many researchers due to its chaotic nature and the ability to replicate similar dynamics in our natural world. This, together with Van der Pol equation has become one of the most common examples of nonlinear oscillations.

Another well studied but still challenging equation in nonlinear dynamics is the combination of the Duffing and the Van der Pol equation. The nonlinearity of the equation makes it difficult to find the first integrals and its solutions. Not much progress was made on the integrability of the Duffing-Van der Pol nonlinear oscillator equation until Chandrasekar et.al [15] have established complete integrability of this equation and have derived a general solution for a specific choice of the arbitrary parameters.

In this paper we have introduced a generalised Van der Pol equation with a new form of potential which exhibits integrability. An analytic solution of the above mentioned system is also derived using the Chiellini integrability condition.

The application of the Chiellini integrability condition to find the solutions of nonlinear differential equations has been recently promoted by two groups; Harko, Mak and their coauthors [16, 17, 18, 20] and Mancas and Rosu [21, 22, 23]. It must be worth to note that the Chiellini integrability condition appears quite naturally for Hamiltonization of the Liénard equation using the Jacobi multiplier technique. Our intention in this letter is to extend the scope of the integrability condition given by Chiellini to find out solutions to some new kind of nonlinear systems. In particular, we obtain the analytic solutions of the damped Ermakov-Painlevé II equation, generalized damped Milne-Pinney equation and

generalised Van der Pol equation.

2. CHIELLINI INTEGRABILITY CONDITION AND ITS APPLICATION

The first order Abel differential equation [7]of the first kind plays an important role in many physical and mathematical problems. The connection between the second-order nonlinear differential equations and the Abel equation is well known [8],[9] and the solutions to such differential equations can often be obtained via the solutions of the corresponding Abel differential equations. A second order differential equation of the Lie´nard type [10] given by

$$\ddot{Y} + g(Y)\dot{Y} + h(Y) = 0 \tag{2.1}$$

may be transformed into a first-order Abel differential equation of second kind, namely

$$z\frac{dz}{dY} + g(Y)z + h(Y) = 0 \tag{2.2}$$

by the transformation $\dot{Y} = z(Y(t))$ which in turn is transformed to the Abel equation of first kind

$$\frac{dX}{dY} = g(Y)X^2 + h(Y)X^3 \tag{2.3}$$

Via the transformation $z = \frac{1}{X}$. However, the criterion of integrability of such equations greatly depends on the expressions of $g(Y)$ and $h(Y)$. An important observation by Chiellini [11] in 1931 states that a first kind Abel differential equation (2.3) is exactly integrable if the functions $g(Y)$ and $h(Y)$ satisfies the condition

$$\frac{d}{dY}\left(\frac{h(Y)}{g(Y)}\right) = pg(Y) \tag{2.4}$$

For some constant p. This integrability condition has been applied in 1960s by Bandic´ who wrote a couple of mathematical papers [12],[13] and then by Borghero and Melis[14] in the Szebehely's problem. Recently, this integrability condition as gained much attention by Mak and Harko [16],[17],[18] in obtaining general solutions of the first-kind Abel equations from a particular solution. This result has also been used by Yurov and Yurov [19] in cosmology and again by Harko et al [20] incase of particular Lie´nard equations.

The Chiellini condition, not only ensures the integrability of a system, but it also helps to find the solution. If we further require that $z = c_k \frac{h(Y)}{g(Y)}$ then its substitution in equation (2.2) leads to

$$pc_k^2 + c_k + 1 = 0 \Rightarrow c_k = \frac{-1 \pm \sqrt{1-4p}}{2p}$$

For simplicity, we choose $c_k = 1$ which gives the value of $p = -2$ in equation (2.4). Thus, from

$\dot{Y} = z(Y(t))$ we have

$$\dot{Y} = \frac{h(Y)}{g(Y)}. \tag{2.5}$$

Using this result we arrive at a much relevant observation that equation (2.1) can be turned to the non dissipative equation

$$\ddot{Y} + H(Y) = 0, \quad H(Y) = 2h(Y) \tag{2.6}$$

where the function $h(Y)$ is scaled up by a factor 2.

This result allows us to find the dissipation function in (2.1) without actually knowing Y. Multiplying $\ddot{Y} + 2h(Y) = 0$ by $\dot{Y}$ and integrating we have

$$\dot{Y}\ddot{Y} + 2\dot{Y}h(Y) = 0 \Rightarrow \dot{Y}^2 = -4\int h(Y)dY + c \tag{2.7}$$

Where c is an integrating constant. Thus, from (2.5) we have

$$g(Y) = \frac{h(Y)}{\sqrt{c - 4\int h(Y)d(Y)}} \tag{2.8}$$

Upon further integration of (2.7)we have

$$t - t_0 = \int \frac{dY}{\sqrt{c - 4\int h(Y)d(Y)}} \qquad \text{where } t_0 \text{ depends on initial conditions.} \tag{2.9}$$

2.1. Dissipative Ermakov- Painlevé II Equation

Combining the terms of both Ermakov-Pinney equation and the Painlevé II we obtain the following equation

$$\ddot{y} + \frac{\tau}{2}y + \varepsilon y^3 = -\frac{1}{4y^3}\left(\gamma - \frac{\varepsilon}{2}\right)^2 \tag{2.10}$$

This nonlinear equation is known as the (single component) Ermakov- Painlevé II equation and was derived by Rogers etal [24,6]. It is related [25] to the Painlevé II equation

$$\ddot{z} = 2z^3 + \tau z + \gamma, \tag{2.11}$$

where $z = \frac{\varepsilon}{2y^2}\left(\gamma - \frac{\varepsilon}{2} - 2y\dot{y}\right)$

If we express $y = \sqrt{|\phi|^2 + |\psi|^2}$ then the canonical single component Ermakov- Painlevé II equation yields a particular Ermakov-Ray-Reid system (for details, see[6])and admits the characteristic invariant which may be exploited systematically to construct the solutions.

Let us assume the equation (2.10) as

$$\ddot{y}+\lambda y+\varepsilon y^3=\frac{\eta}{y^3} \tag{2.12}$$

Where $\lambda=\frac{\tau}{2}$ and $\eta=-\frac{1}{4}\left(\gamma-\frac{\varepsilon}{2}\right)^2$

If $h(y)=\lambda y+\varepsilon y^3-\frac{\eta}{y^3}$ (2.13)

then equation (2.12) can be written as

$$y\ddot{y}+h(y)=0 \tag{2.14}$$

We introduce the dissipative Ermakov- Painlevé II equation having same h(y) as in the non dissipative case but with an additional damping term. The equation

$$\ddot{Y}+g(Y)\dot{Y}+h(Y)=0 \tag{2.15}$$

is called Chiellini dissipative Ermakov Painlevé II equation because the damping coefficient g(Y) will be obtained from the Chiellini integrability condition. Using h(Y) from (2.13) in (2.7), we have

$$\dot{Y}=\sqrt{c-\varepsilon Y^4-2\lambda Y^2-2\eta Y^{-2}} \tag{2.16}$$

and in (2.8), we have

$$g(Y)=\frac{\lambda Y^2+\varepsilon Y^4-\eta Y^{-2}}{\sqrt{-\varepsilon Y^6-2\lambda Y^4+cY^2-2\eta}} \tag{2.17}$$

Further from (2.16) upon integration once more, we have

$$Y^2=\begin{cases}\dfrac{1}{(t-t_0)^2}+\sqrt{\dfrac{c}{3}} & \in=-1\\ \left(\dfrac{c^2}{16\lambda^2}-\dfrac{\eta}{\lambda}\right)^{1/2}\sin\left(2\sqrt{2}(t-t_0)\right)+\dfrac{c}{4\lambda} & \in=\ 0\end{cases} \tag{2.18}$$

t_0 depending on initial conditions.

2.2. Generalized Dissipative Milne-Pinney Equation

At first we embark a simple equation of this category and obtain its solution. Let us consider the equation

$$\ddot{Y}+g(Y)\dot{Y}-\frac{\delta}{Y^5}=0 \tag{2.19}$$

This may be written as equation (2.1) with $h(Y)=-\frac{\delta}{Y^5}$.

Following similar arguments we have $\dot{Y} = \sqrt{c - \delta Y^{-4}}$ (2.20)

The dissipative term $g(Y) = -\dfrac{\delta}{Y^3\sqrt{cY^4 - \delta}}$ and a parametric solution to the equation (2.20) in terms of Weierstrass $\wp$ function is given as

$$t = y_0^2\omega + \left[-\frac{y_0 f'(y_0)}{2\wp'(\omega_0)} + \frac{f'(y_0)^2\wp''(\omega_0)}{16\wp'(\omega_0)^3}\right]\log\frac{\sigma(\omega+\hat{c}+\omega_0)}{\sigma(\omega+\hat{c}-\omega_0)} - \frac{f'(y_0)^2}{16\wp'(\omega_0)^2}[\zeta(\omega+\hat{c}+\omega_0)+\zeta(\omega+\hat{c}-\omega_0)]$$
$$+(\omega+\hat{c})\left(\frac{y_0 f'(y_0)}{\wp'(\omega_0)}\zeta(\omega_0) - \frac{f'(y_0)^2}{16}\left[\frac{2\wp(\omega_0)}{\wp'(\omega_0)^2} + \frac{2\wp''(\omega_0)\zeta(\omega_0)}{\wp'(\omega_0)^3}\right]\right) + \hat{\delta} \tag{2.21}$$

and

$$Y = y_0 + \frac{f'(y_0)}{4\left[\wp(\omega+\hat{c}) - \dfrac{f''(y_0)}{24}\right]} \tag{2.22}$$

where $\hat{\delta}$ is an integrating constant and $\hat{c}$ being any fixed constant.

Also $f(Y) = cY^4 - \delta$, y_0 is a root of the equation $f(Y) = 0$ and $\wp(\omega) = \wp(\omega, g_2, g_3)$ is the *Weierstrass* $\wp$ - *function* attached to the *Weierstrass Invariants*. Here $g_2 = 3\alpha_2^2 - 4\alpha_1\alpha_3$ and $g_3 = 2\alpha_1\alpha_2\alpha_3 - \alpha_2^3 - \alpha_0\alpha_3^2$; $\alpha_0 = c$, $\alpha_1 = cy_0$, $\alpha_2 = cy_0^2$, $\alpha_3 = cy_0^3$. $\sigma(\omega)$ and $\eta(\omega)$ are the Weierstrass sigma and Weierstrass zeta functions respectively, $\wp(\omega_0) = \dfrac{f''(y_0)}{24}$ (for a choice of ω_0).

Generalizing the results of equation (2.1), we consider the following example of generalized damped Milne-Pinney equation. Let us consider the equation

$$\ddot{Y} + g(Y)\dot{Y} + \lambda Y = \frac{k_1}{Y^3} + \frac{k_2}{Y^2} + \sum_{n=0}^{R}\delta_n Y^{2n+1} \tag{2.23}$$

For $R = 0$ we have $\dot{Y} = \sqrt{2(\delta_0 - \lambda)Y^2 + c - 4k_2Y^{-1} - 2k_1Y^{-2}}$ (2.24)

The dissipative term $g(Y) = \dfrac{(\lambda - \delta_0)Y - k_2Y^{-2} - k_1Y^{-3}}{\sqrt{2(\delta_0 - \lambda)Y^2 + c - 4k_2Y^{-1} - 2k_1Y^{-2}}}$

Further integration of (2.24) yields a parametric solution in terms of Weierstrass $\wp$ function given as

$$t = y_0\omega + \frac{f'(y_0)}{4\wp'(\omega_0)}\left[\log\frac{\sigma(\omega+\hat{c}+\omega_0)}{\sigma(\omega+\hat{c}-\omega_0)} + 2(\omega+\hat{c})\zeta(\omega_0)\right] + \delta \tag{2.25}$$

$$Y = y_0 + \frac{f'(y_0)}{4\left[\wp(\omega+\hat{c}) - \dfrac{f''(y_0)}{24}\right]} \tag{2.26}$$

Where δ is an integrating constant and $\hat{c}$ being any fixed constant.

Also $f(Y) = 2(\delta_0 - \lambda)Y^4 + cY^2 - 4k_2Y - 2k_1$, y_0 is a root of the equation $f(Y) = 0$ and $\wp(\omega) = \wp(\omega, g_2, g_3)$ is the *Weierstrass* $\wp$ - *function*. Here $g_2 = 3\alpha_2^2 - 4\alpha_1\alpha_3$ and $g_3 = 2\alpha_1\alpha_2\alpha_3 - \alpha_2^3 - \alpha_0\alpha_3^2$;

$$\alpha_0 = 2(\delta_0 - \lambda)$$
$$\alpha_1 = 2(\delta_0 - \lambda)y_0$$
$$\alpha_2 = 2(\delta_0 - \lambda)y_0^2 + \frac{c}{6}$$
$$\alpha_3 = 2(\delta_0 - \lambda)y_0^3 + 3\frac{cy_0}{6} - k_2$$

$\wp(\omega_0) = \dfrac{f''(y_0)}{24}$ (for a choice of ω_0)is not equal to any of the roots of $4y^3 - g_2y - g_3 = 0$

2.3. Generalised Van Der Pol Equation with New Form of Potential

Let us consider the nonlinear differential equation

$$\ddot{Y} + \left(\alpha + \beta Y^2 + \gamma Y\right)\dot{Y} + h(Y) = 0 \tag{2.27}$$

This equation may be written as $\ddot{Y} + g(Y)\dot{Y} + h(Y) = 0$ (2.28)

Where, $g(Y) = \alpha + \beta Y^2 + \gamma Y$ and $h(Y)$ is a general function of $Y(t)$. Integrating (2.4) with $p = -2$ we have $h(Y) = g(Y) - 2\int g(Y)dY + c$. (2.29)

Where c is an integrating constant. From (2.29) for $g(Y) = \alpha + \beta Y^2 + \gamma Y$ we have

$$h(Y) = -\frac{2}{3}\beta^2Y^5 + \frac{5}{3}\beta\gamma Y^4 - (\gamma^2 + \frac{8}{3})Y^3 + (c\beta - 3\alpha\gamma) + (c\gamma - 2\alpha^2)Y + c\alpha \tag{2.30}$$

And from (2.5) we have $\dfrac{dY}{dt} = c - 2\alpha Y - \gamma Y^2 - \dfrac{2}{3}Y^3$ (2.31)

Integrating once we have

$$t_0 - t = \frac{3}{2\beta\left[3\left(\Gamma^2 + \Phi^2\right) + \frac{\gamma}{2\beta}(\Gamma + \Phi) + \frac{3\gamma^2}{4\beta^2} - \frac{3\alpha}{\beta}\right]}[\ln\frac{Y - \Gamma - \Phi + \frac{\gamma}{2\beta}}{\sqrt{Y^2 + BY + \hat{C}}} - \frac{3(\Gamma + \Phi)}{\sqrt{3\left(\Gamma^2 + \Phi^2\right) + \frac{2\gamma}{\beta}(\Gamma + \Phi) - \frac{3\gamma^2}{2\beta^2} + \frac{6\alpha}{\beta}}}\tan^{-1}[\frac{2Y + B}{\sqrt{3\left(\Gamma^2 + \Phi^2\right) + \frac{2\gamma}{\beta}(\Gamma + \Phi) - \frac{3\gamma^2}{2\beta^2} + \frac{6\alpha}{\beta}}}]] \tag{2.32}$$

where $B = \Gamma + \Phi + \dfrac{\gamma}{\beta}$, $\quad \hat{C} = \Gamma^2 + \Phi^2 + \dfrac{\gamma}{\beta}\left(\Gamma + \Phi\right) + \dfrac{\alpha}{\beta}$

and

$$2\Gamma^3 = \left(\frac{-\gamma^3}{4\beta^3} + \frac{3\alpha\gamma}{2\beta^2} + \frac{3c}{2\beta}\right) + \sqrt{\left(\frac{-\gamma^3}{4\beta^3} + \frac{3\alpha\gamma}{2\beta^2} + \frac{3c}{2\beta}\right)^2 - 4\left(\frac{\gamma^2}{4\beta^2} - \frac{\alpha}{\beta}\right)^3}$$

$$2\Phi^3 = \left(\frac{-\gamma^3}{4\beta^3} + \frac{3\alpha\gamma}{2\beta^2} + \frac{3c}{2\beta}\right) - \sqrt{\left(\frac{-\gamma^3}{4\beta^3} + \frac{3\alpha\gamma}{2\beta^2} + \frac{3c}{2\beta}\right)^2 - 4\left(\frac{\gamma^2}{4\beta^2} - \frac{\alpha}{\beta}\right)^3}$$

3. CONCLUSIONS

In this letter, we have generated some second order nonlinear damped equations using the Chiellini condition of integrability. We have also shown that they are exactly integrable and have computed the solutions. If the coefficients of the second-order nonlinear equations satisfy some specific conditions that follow from the Chiellini integrabilty condition, then the general solution of such equations can be obtained in an exact parametric form. Moreover, the solution to the generalized VanderPol equation are obtained for arbitrary values of α, β and γ.

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